

L^p -LIOUVILLE THEOREMS ON COMPLETE SMOOTH METRIC MEASURE SPACES

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ABSTRACT. We study some function-theoretic properties on complete smooth metric measure space $(M, g, e^{-f} dv)$ with its Bakry-Émery Ricci curvature bounded below. Assuming the bound of function f , we derive a Moser's parabolic Harnack inequality of the f -heat equation, which leads to upper and lower Gaussian bounds on the f -heat kernel. We also prove various L^p -Liouville theorems in terms of the lower bound of Bakry-Émery Ricci curvature and the bound of function f , which generalize the classical Ricci curvature case and the N -Bakry-Émery Ricci curvature case.

1. INTRODUCTION AND MAIN RESULTS

1.1. Background. Let (M, g) be an n -dimensional complete noncompact Riemannian manifold and f be a smooth function on M . We define a symmetric diffusion operator Δ_f (or f -Laplacian), which is given by

$$\Delta_f := \Delta - \nabla f \cdot \nabla,$$

where Δ and ∇ are the Laplacian and covariant derivative of the metric g . f -Laplacian Δ_f is the infinitesimal generator of the Dirichlet form

$$\mathcal{E}(\phi_1, \phi_2) = \int_M (\nabla \phi_1, \nabla \phi_2) d\mu, \quad \forall \phi_1, \phi_2 \in C_0^\infty(M),$$

where μ is an invariant measure of Δ_f given by $d\mu = e^{-f} dv$, and where dv is the volume element induced by the metric g . It is well-known that Δ_f is self-adjoint with respect to the weighted measure $d\mu$. The triple $(M, g, e^{-f} dv)$ is customarily called a complete smooth metric measure space. On this measure space, we often consider the f -heat equation

$$(\partial_t - \Delta_f)u = 0$$

instead of the classical heat equation. If the function u is independent of time t , then u is the f -harmonic function. In this paper, we denote $H(x, y, t)$ be the f -heat kernel. That is, for each $x \in M$, $H(x, y, t) = u(y, t)$ is the minimal solution of the f -heat equation with $u(0, y) = \delta_x(y)$. Equivalently, $H(x, y, t)$ is the kernel of the semigroup $P_t = e^{t\Delta_f}$ associated to the Dirichlet form $\mathcal{E}(\phi, \phi)$.

On the metric measure space $(M, g, e^{-f} dv)$, following Bakry and Émery [1] and [2] (see also [22] and [24]), we define the Bakry-Émery Ricci curvature as follows:

$$Ric_f := Ric + Hess(f),$$

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where Ric denotes the Ricci curvature of the manifold and $Hess$ denotes the Hessian with respect to the manifold metric. The Bakry-Émery Ricci curvature is a natural extension of Ricci curvature. If f is constant, Ric_f returns to the Ricci curvature Ric . The Bakry-Émery Ricci curvature has been extensively studied because it often shares similar properties of the ordinary Ricci curvature. For example, there exists an interesting Bochner type identity

$$\Delta_f |\nabla u|^2 = 2|Hess(u)|^2 + 2\langle \nabla u, \nabla \Delta_f u \rangle + 2Ric_f(\nabla u, \nabla u).$$

This identity is parallel to the Ricci curvature case, which plays an important role in studying comparison theorems (see [39]). For more extended results, the interested reader can consult [3], [4], [8], [13], [23], [24], [25], [35], [36], [38] and [40].

Also, the Bakry-Émery Ricci curvature has become an important object of study in the geometric analysis, in large part due to the so-called gradient Ricci soliton. Recall that a complete manifold (M, g) is a gradient Ricci soliton if

$$Ric_f = \rho g$$

holds for some function f and constant ρ . The soliton is called expanding, steady and shrinking, accordingly, if $\rho < 0$, $\rho = 0$ and $\rho > 0$. Ricci soliton of manifolds possess many interesting geometric and topological properties. See, for example, [6], [7] and [29] for nice explanations on this subject.

Recently, there has been renewed interest in the Bakry-Émery Ricci curvature. For example, G. Catino, C. Mantegazza, L. Mazzieri and M. Rimoldi [9], P. Peterson and W. Wylie [30], and S. Pigola, M. Rimoldi and A.G Setti [31] established various Liouville-type or rigid theorems on this curvature. Prior to their works, X.-D. Li [22] studied the L^1 -Liouville theorem if the N -Bakry-Émery Ricci curvature is bounded below by a negative quadratic polynomial of the distance function, which is an extension of the classical L^1 -Liouville theorem on Ricci curvatures, proved by Peter Li [18]. Here the N -Bakry-Émery Ricci curvature is defined by

$$Ric_f^N := Ric + Hess(f) - \frac{df \otimes df}{N},$$

where N is a positive constant. However, as X.-D. Li said in Subsection 8.6 of [22], we cannot prove the L^1 -Liouville theorem if we only keep the lower bound of the Bakry-Émery Ricci curvature to be a negative quadratic polynomial of the distance function. Because we can not obtain the Li-Yau type parabolic Harnack inequality under only this curvature assumption. Now it is natural to pose the following problem: What is the optimal geometric and analytic condition on the metric measure space such that the Li-Yau parabolic Harnack inequality holds?

In recent papers [27, 28], Munteanu and Wang partially answered the above question. In particular, they derived gradient estimates and Liouville properties for positive f -harmonic functions under some growth assumption on f . Their theorems take the form of Yau's classical result on the positive f -harmonic functions, but their proof is new and quite different in spirit from Yau's direct Bochner formula method [42]. Their proof approach essentially relies on the well-known De Giorgi-Nash-Moser theory, which motivates our proof of Theorem 1.1 in Subsection 1.2.

1.2. Main results. The purpose of this paper is to study some geometric inequalities for the f -heat equation and various L^p -Liouville theorems for the f -harmonic function on complete smooth metric measure space $(M, g, e^{-f} dv)$ with its Bakry-Émery Ricci curvature bounded below, which is divided into two parts.

In the first part, borrowing the idea of Munteanu and Wang in [27, 28], we will study some geometric inequalities, such as Harnack inequalities, Hölder continuity estimates, f -heat kernel estimates on complete smooth metric measure spaces. We first present a parabolic Harnack inequality of the f -heat equation on complete smooth metric measure spaces.

Theorem 1.1. *Let $(M, g, e^{-f} dv)$ be an n -dimensional complete noncompact smooth metric measure space. If $\text{Ric}_f \geq -(n-1)K$ and $|f|(x) \leq A$ for some nonnegative constants K and A , then there exist a constant $c(n, A)$ such that, for any ball $B_o(r)$, $o \in M$, $0 < r < R \leq \infty$ and for any smooth positive solution u of the f -heat equation in the cylinder $Q = B_o(r) \times (s - r^2, s)$,*

$$\sup_{Q_-} \{u\} \leq e^{c(n,A)(1+Kr^2)} \cdot \inf_{Q_+} \{u\},$$

where $Q_- := B_o(\frac{1}{2}r) \times (s - \frac{3}{4}r^2, s - \frac{1}{2}r^2)$ and $Q_+ := B_o(\frac{1}{2}r) \times (s - \frac{1}{4}r^2, s)$.

The sketch proof of Theorem 1.1 will be given in Section 2. Its proof follows the Moser iteration technique [26], which involves a local Sobolev inequality on a smooth metric measure space. Munteanu and Wang [28] used similar technique to derive an elliptic Harnack inequality for f -harmonic function. When the metric measure space is a Riemannian manifold, that is function f is constant, this case was obtained independently by L. Saloff-Coste [32] and A. Grigor'yan [14].

A standard consequence of Theorem 1.1 is a strong Liouville theorem for any f -harmonic function (see Corollary 3.2 in Section 3). Moreover, Theorem 1.1 also implies two-sided f -heat kernel bounds with respect to smooth metric measure spaces. This result is essential parallel to the case of heat equation on Riemannian manifolds, obtained by L. Saloff-Coste [34].

Theorem 1.2. *Let $(M, g, e^{-f} dv)$ be an n -dimensional complete noncompact smooth metric measure space. If $\text{Ric}_f \geq -(n-1)K$ and $|f|(x) \leq A$ on the ball $B_o(2R)$ for some nonnegative constants K and A , then there exist positive constants c_i , $i = 5, 6, 7, 8$, depending only on n and A such that*

$$\frac{e^{-c_6(1+Kt)}}{V_f(B_x(\sqrt{t}))} \exp\left(-c_5 \frac{d^2(x, y)}{t}\right) \leq H(x, y, t) \leq \frac{e^{c_8(1+Kt)}}{V_f(B_x(\sqrt{t}))} \exp\left(-c_7 \frac{d^2(x, y)}{t}\right)$$

for any $x, y \in B_o(R/2)$ and $0 < t < R^2/4$, where $V_f(B_x(\sqrt{t}))$ denotes the f -volume of the ball $B_x(\sqrt{t})$ with respect to $e^{-f} dv$.

Remark 1.3. Theorem 1.2 gives a detailed description of the coefficients of two-sided Gaussian bounds on the f -heat kernel. This estimate has an important application in the proof of the following Theorem 1.6.

We point out that the proof of Theorem 1.2 is different from the classical Li-Yau trick [18]. In [18], two-sided Gaussian bounds on the heat kernel are obtained by the Li-Yau gradient estimate. However, in our case it seems to be impossible to utilize Li-Yau gradient estimate method directly to derive upper and lower bounds on the f -heat kernel on complete smooth metric measure spaces. In our setting, two-sided Gaussian bounds on f -heat kernel strongly rely on the Moser's parabolic Harnack inequality and the integral estimate of the f -heat kernel due to E. B. Davies [11], which are similar to the arguments of L. Saloff-Coste [32, 33, 34] and A. Grigor'yan [14]. Please see the Section 4 for a detailed discussion.

In the second part of this paper, we will investigate various L^p -Liouville theorems for f -harmonic functions on complete noncompact metric measure space $(M, g, e^{-f} dv)$ under different assumptions on Ric_f and f .

We first start with a L^p -Liouville theorem for positive f -subharmonic functions when $1 < p < \infty$.

Theorem 1.4. *Let $(M, g, e^{-f} dv)$ be an n -dimensional complete smooth metric measure space. For any $1 < p < \infty$, there does not exist any nonconstant, nonnegative, $L^p(\mu)$ -integrable f -subharmonic function. This constant must be zero if M has infinite f -volume.*

The proof of this result is almost the same as the arguments of the classical case [43], which will be given in Subsection 6.1. Note that this theorem holds without any curvature condition.

We also deal with the L^p -Liouville theorem in case of $0 < p < 1$. This case is parallel to the case of subharmonic functions on a manifold with Ricci curvature, discussed by Li and Schoen [20]. See Subsection 6.2 for the detailed discussion.

Theorem 1.5. *Let $(M, g, e^{-f} dv)$ be an n -dimensional complete noncompact smooth metric measure space. If*

$$Ric_f \geq -\delta(n)(1 + r(x))^{-2},$$

where $r(x)$ is the distance to a fixed point $o \in M$, and f is bounded, then any nonnegative L^p -integrable ($0 < p < 1$) f -subharmonic function must be identically constant. Moreover, this constant must be zero if M has infinite f -volume.

Finally, motivated by the P. Li's work [18] and X.-D. Li's generalization [22], we obtain a new L^1 -Liouville theorem on the smooth metric measure space.

Theorem 1.6. *Let $(M, g, e^{-f} dv)$ be an n -dimensional complete noncompact smooth metric measure space. Assume that*

$$Ric_f \geq -C(1 + r^2(x))$$

for some constant $C > 0$, and f is bounded. Then any nonnegative $L^1(\mu)$ -integrable f -subharmonic function must be identically constant. Moreover, this constant must be zero if M has infinite f -volume.

Theorem 1.6 partially answers a question posed by X.-D. Li (see Subsection 8.6 in [22]). Its proof is similar to the arguments of [18], where a critical step is the usage of the upper Gaussian bound on the f -heat kernel (Theorem 1.2). The detailed discussion shall be carried in Subsection 6.3.

Remark 1.7. We remark that the absolute value of a f -harmonic function is a nonnegative f -subharmonic. Therefore we can conclude that a complete metric measure space does not admit any nonconstant $L^p(\mu)$ -integrable f -harmonic function under the same hypotheses of Theorems 1.4, 1.5 and 1.6, respectively.

Remark 1.8. As many recent authors said in [12], [37] and [41], if the condition on f bounded is replaced by $|\nabla f|$ bounded, then similar results of Theorems 1.5 and 1.6 can be immediately obtained by modifying the arguments of [22]. Indeed, the conditions $Ric_f \geq -(n-1)K$ and $|\nabla f| \leq a$ imply that

$$Ric_f^N \geq -(n-1) \left(K + \frac{a^2}{N(n-1)} \right).$$

In the end of this section, we would like to say that our basic idea of proving L^p -Liouville theorems was inspired by the classical work [43], [20], [18] and [22], but we need to do a lot of work in order to develop this basic idea in our situation. For example, an upper bound of the f -heat kernel is quite essential in proving the L^1 -Liouville theorem for f -harmonic functions. However, to the author's knowledge, it seems to be impossible to directly apply Li-Yau gradient estimate to derive the upper bound on the f -heat kernel under our curvature assumptions. Fortunately, we can apply Moser's parabolic Harnack inequality to derive it.

The rest of this paper is organized as follows. In Section 2, we recall the Neumann Poincaré inequality and the local Sobolev inequality on complete smooth metric measure spaces. After that, we establish a Moser's version of parabolic Harnack inequality. In Section 3, using the parabolic Harnack inequality, we obtain a quantitative Hölder continuity estimate for a solution to the f -heat equation, which implies a strong Liouville theorem. In Section 4, we prove the two-sided Gaussian bounds on the f -heat kernel on complete smooth metric measure spaces. In Section 5, we prove a mean value inequality on complete smooth metric measure spaces, which is similar to the case of harmonic functions on a manifold, obtained by Li and Schoen [20]. In Section 6, we derive L^p -Liouville theorems of f -Laplacian on complete smooth metric measure spaces by following the ideas in [18], [20] and [43].

2. POINCARÉ, SOBOLEV AND HARNACK INEQUALITIES

Let $\Delta_f = \Delta - \nabla f \cdot \nabla$ be an f -Laplacian with an invariant measure $d\mu = e^{-f} dv$ on a complete Riemannian manifold. For a set Ω , we will denote by $V(\Omega)$ the volume of Ω with respect to the usual volume form dv , and $V_f(\Omega)$ the f -volume of Ω with respect to $e^{-f} dv$. Throughout of this section, we will assume

$$\text{Ric}_f \geq -(n-1)K \quad \text{and} \quad |f|(x) \leq A$$

for some nonnegative constants K and A . Under these assumptions, we have the Laplacian and volume comparison results with respect to the smooth metric measure space $(M, g, e^{-f} dv)$.

Lemma 2.1 (Wei and Wylie [39]). *Let $(M, g, e^{-f} dv)$ be an n -dimensional complete noncompact smooth metric measure space. If $\text{Ric}_f \geq -(n-1)K$ and $|f|(x) \leq A$ for some nonnegative constants K and A , then*

$$\Delta_f r(x, y) \leq (n-1+4A)\sqrt{K} \coth \sqrt{K} r.$$

for any $0 < r < R$. Hence along any minimizing geodesic starting from $x \in B_o(R)$ we have

$$(2.1) \quad \frac{V_f(B_x(r_2))}{V_f(B_x(r_1))} \leq \frac{V_K^{n+4A}(r_2)}{V_K^{n+4A}(r_1)}$$

for any $0 < r_1 < r_2 < R$. Here $V_K^{n+4A}(r)$ be the volume of the radius r -ball in the model space M_K^{n+4A} , the simply connected model space of dimension $n+4A$ with constant curvature K .

From (2.1), we easily deduce that

$$(2.2) \quad V_f(B_x(2r)) \leq 2^{n+4A} e^{C(n,A)\sqrt{K}r} \cdot V_f(B_x(r))$$

for any $0 < r < R$. This inequality possesses the volume doubling property, which plays a crucial role in our paper. Recall that a manifold admits a volume doubling property if for any fixed $0 < R \leq \infty$, there exists a constant C such that

$$V_f(B_x(2r)) \leq C \cdot V_f(B_x(r))$$

for any $0 < r < R$ and $x \in M$. By Lemma 2.1, following the L. Saloff-Coste's proof (see Theorem 5.6.5 in [34]) step by step, we can easily get a Neumann Poincaré inequality with respect to smooth metric measure spaces.

Lemma 2.2. *Let $(M, g, e^{-f} dv)$ be an n -dimensional complete noncompact smooth metric measure space. If $\text{Ric}_f \geq -(n-1)K$ and $|f|(x) \leq A$ for some nonnegative constants K and A , then for any $x \in B_o(R)$*

$$(2.3) \quad \int_{B_o(r)} |\varphi - \varphi_{B_o(r)}|^2 e^{-f} dv \leq e^{c_1(1+\sqrt{K}r)} \cdot r^2 \int_{B_o(r)} |\nabla \varphi|^2 e^{-f} dv$$

for all $0 < r < R$ and $\varphi \in C^\infty(B_o(r))$, where $\varphi_{B_o(r)} := V_f^{-1}(B_o(r)) \int_{B_o(r)} \varphi e^{-f} dv$. The constant c_1 depends only on the dimension n and A .

Remark 2.3. Munteanu and Wang [27] proved the above Poincaré inequality under a weaker hypothesis on the oscillation of f on unit balls based on the arguments in [5]. That is, (2.3) holds on the *unit* balls if $\text{Ric}_f \geq -(n-1)K$ and $\sup_{y \in B_x(1)} |f(y) - f(x)| \leq a$ for any $x \in M$. But in our case, the Poincaré inequality can hold on *any* radius of balls due to a stronger assumption on f , which is a crucial step on the proof of L^1 -Liouville theorem. Because in the course of proof of L^1 -Liouville result, we need to let the radius of balls tend to infinity. Also note that when f is constant, (2.3) was confirmed by L. Saloff-Coste (see (6) in [33] or Theorem 5.6.5 in [34]).

Combining Lemma 2.1, Lemma 2.2 and the argument in [32], we have a local Sobolev inequality, which is one of the key technical points needed to apply Moser's iterative technique to derive parabolic Harnack inequalities for the f -heat equation.

Lemma 2.4. *Let $(M, g, e^{-f} dv)$ be an n -dimensional complete noncompact smooth metric measure space. If $\text{Ric}_f \geq -(n-1)K$ and $|f|(x) \leq A$ for some nonnegative constants K and A , then for any constant $p > 2$, there exist a constant c_2 , depending on n and A such that*

$$\left(\int_{B_o(r)} |\varphi|^{\frac{2p}{p-2}} e^{-f} dv \right)^{\frac{p-2}{p}} \leq \frac{e^{c_2(1+\sqrt{K}r)} \cdot r^2}{V_f(B_o(r))^{\frac{2}{p}}} \int_{B_o(r)} (|\nabla \varphi|^2 + r^{-2} |\varphi|^2) e^{-f} dv$$

for all $0 < r < R$ and $\varphi \in C_0^\infty(B_o(r))$.

Sketch proof of Lemma 2.4. The proof is nearly same as that of Theorem 2.1 in [32] or Theorem 3.1 in [33] except our discussion with respect to the weighted measure $e^{-f} dv$. When f is constant, this result was confirmed by L. Saloff-Coste [32] (see also Theorem 3.1 in [33]). We refer the reader to these papers for the nice proof. \square

Remark 2.5. In Lemma 2.4, the local Sobolev inequality is slight different from Munteanu and Wang's Neumann Sobolev inequality (Lemma 3.3 in [27]). Here we mainly follow the arguments of L. Saloff-Coste [32] to derive the local Sobolev inequality, whereas Munteanu and Wang proved their local Neumann Sobolev inequality based on the argument in [16]. Also note that our local Sobolev inequality holds on *any* radius of balls due to a stronger assumption on f . But Munteanu and

Wang [27] established the local Neumann Sobolev inequality only on the *unit* balls due to a weaker hypothesis on the oscillation of f on unit balls.

Below we shall present a result concerning the Harnack inequality for the f -heat equation, which is very much parallel to the case that f is constant, obtained by L. Saloff-Coste [32] and A. Grigor'yan [14].

Theorem 2.6. *Let $(M, g, e^{-f} dv)$ be an n -dimensional complete noncompact smooth metric measure space. Fix $0 < R \leq \infty$. Assume that volume doubling property (2.2) and the Neumann Poincaré inequality (2.3) are satisfied for this R . Then there exist constants c_3 depending on n and A such that, for any ball $B_o(r)$, $o \in M$, $0 < r < R$ and for any smooth positive solution u of the f -heat equation in the cylinder $Q = B_o(r) \times (s - r^2, s)$, we have*

$$\sup_{Q_-} \{u\} \leq e^{c_3(1+Kr^2)} \cdot \inf_{Q_+} \{u\},$$

where $Q_- := B_o(\frac{1}{2}r) \times (s - \frac{3}{4}r^2, s - \frac{1}{2}r^2)$ and $Q_+ := B_o(\frac{1}{2}r) \times (s - \frac{1}{4}r^2, s)$.

Sketch proof of Theorem 2.6. The proof is the same as the arguments of [32] or [33] except our discussion with respect to the weighted measure. Indeed this result follows from the standard Moser's technique. Since the conditions of Theorem 2.6 imply a family of local Sobolev inequalities due to Lemma 2.4, combining the volume doubling property, this is enough to run the Moser's iteration procedure to prove Theorem 2.6, as explained in [32] or [33]. \square

Combining Lemmas 2.1, 2.2 and Theorem 2.6, we immediately have that:

Corollary 2.7. *Let $(M, g, e^{-f} dv)$ be an n -dimensional complete noncompact smooth metric measure space. If $\text{Ric}_f \geq -(n-1)K$ and $|f|(x) \leq A$ for some nonnegative constants K and A , then there exist a constant $c(n, A)$ such that, for any $0 < R \leq \infty$ and ball $B_o(r)$, $o \in M$, $0 < r < R$ and for any smooth positive solution u of the f -heat equation in the cylinder $Q = B_o(r) \times (s - r^2, s)$, we have*

$$\sup_{Q_-} \{u\} \leq e^{c(n, A)(1+Kr^2)} \cdot \inf_{Q_+} \{u\},$$

where $Q_- := B_o(\frac{1}{2}r) \times (s - \frac{3}{4}r^2, s - \frac{1}{2}r^2)$ and $Q_+ := B_o(\frac{1}{2}r) \times (s - \frac{1}{4}r^2, s)$.

3. LIOUVILLE THEOREM

In this section, we will apply the parabolic Harnack inequality to obtain a quantitative Hölder continuity estimate for a solution to the f -heat equation. Using this Hölder continuity property, we shall derive a Liouville property under some suitable assumptions on Ric_f and f .

First, we give the Hölder continuity estimate for any solution of the f -heat equation. When f is constant this was established in Theorem 5.4.7 of [34].

Theorem 3.1. *Under the same assumptions of Theorem 2.6, then there exist $\theta = 1 - e^{-c(n, A)(1+Kr^2)} \in (0, 1)$, $\alpha = \log_2 \theta \in (0, 1)$ and $A_\kappa = 4\theta^{-1}(1 - \kappa)^{-\alpha} > 1$, where $\kappa \in (0, 1)$ such that any solution u of the f -heat equation in $Q = B_o(r) \times (s - r^2, s)$,*

$$\sup_{(y, t), (y', t') \in Q_\kappa} \left\{ \frac{|u(y, t) - u(y', t')|}{[|t - t'|^{1/2} + d(y, y')]^\alpha} \right\} \leq \frac{A_\kappa}{r^\alpha} \sup_Q \{ |u| \},$$

where $Q_\kappa := B_o(\kappa r) \times (s - \kappa r^2, s)$.

Proof. The proof is nearly the same as the argument in [26] (see also [34]) which uses the parabolic Harnack inequality. For the reader's convenience, we include a detailed proof of this result. For any non-negative solution v of the f -heat equation in Q , by Theorem 2.6, we have

$$(3.1) \quad \frac{1}{\bar{V}_f(Q_-)} \int_{Q_-} v d\bar{\mu} \leq \max_{Q_-} \{u\} \leq e^{c(n,A)(1+Kr^2)} \min_{Q_+} \{u\},$$

where $Q_- := B_o(\frac{1}{2}r) \times (s - \frac{3}{4}r^2, s - \frac{1}{2}r^2)$ and $Q_+ := B_o(\frac{1}{2}r) \times (s - \frac{1}{4}r^2, s)$, and where $\bar{V}_f(Q_-)$ denotes the volume of Q_- with respect to the space-time volume form $d\bar{\mu}$. Now we let u be a solution, which is not necessary non-negative, and let M_u, m_u be the maximum and minimum of u in Q . Similarly, let M_u^+, m_u^+ be the maximum and minimum of u in Q_+ . Define

$$\mu_u^- := \frac{1}{\bar{\mu}(Q_-)} \int_{Q_-} v d\bar{\mu},$$

where $d\bar{\mu}$ denotes the natural product measure on $R \times M$: $d\bar{\mu} = dt \times d\mu$, and where $d\mu = e^{-f} dv$. Applying (3.1) to the non-negative solutions $M_u - u$, $u - m_u$ yields

$$M_u - \mu_u^- \leq e^{c(n,A)(1+Kr^2)} (M_u - M_u^+)$$

and

$$\mu_u^- - m_u \leq e^{c(n,A)(1+Kr^2)} (m_u^+ - m_u),$$

which imply that

$$(M_u - m_u) \leq e^{c(n,A)(1+Kr^2)} (M_u - m_u) - e^{c(n,A)(1+Kr^2)} (M_u^+ - m_u^+).$$

If we define the oscillations

$$\omega(u, Q) := M_u - m_u \quad \text{and} \quad \omega(u, Q_+) := M_u^+ - m_u^+$$

of u over Q and Q_+ , then

$$(3.2) \quad \omega(u, Q_+) \leq \theta \omega(u, Q),$$

where we assume $e^{c(n,A)(1+Kr^2)} > 1$, and hence $\theta = 1 - e^{-c(n,A)(1+Kr^2)} \in (0, 1)$.

Now we consider $(y, t), (y', t') \in Q_\kappa$. Let

$$\rho = 2 \max\{d(y, y'), \sqrt{t - t'}\}$$

with $t \geq t'$. Then (y', t') belongs to $Q_0 := B_y(\rho) \times (t - \rho^2, t)$. We also define $\rho_i = 2\rho_{i-1}$, $\rho_0 = \rho$ and $Q_i := B_y(\rho_i) \times (t - \rho_i^2, t)$ for all $i \geq 1$. We easily see that

$$(Q_i)_+ = Q_{i-1}.$$

Hence, as long as Q_i is contained in Q , (3.2) yields

$$\omega(u, Q_{i-1}) \leq \theta \omega(u, Q_i) \quad \text{and} \quad \omega(u, Q_0) \leq \theta^i \omega(u, Q).$$

Below, we consider two cases. If $\rho \leq (1 - \kappa)r$, let k be the integer such that

$$2^k \leq (1 - \kappa)r/\rho < 2^{k+1}.$$

Since $(y, t) \in Q_\kappa$, it follows that

$$\begin{aligned} Q_k &= B_y(2^k \rho) \times (t - 4^k \rho^2, t) \\ &\subset B_y((1 - \kappa)r) \times (t - (1 - \kappa)^2 r^2, t) \\ &\subset B_o(r) \times (s - r^2, s) = Q. \end{aligned}$$

Hence we have

$$\omega(u, Q_0) \leq \theta^k \omega(u, Q) \leq \theta^{-1} (1 - \kappa)^{-\alpha} \left(\frac{\rho}{r}\right)^\alpha \omega(u, Q)$$

with $\alpha = \log_2 \theta$. This implies

$$\frac{|u(y, t) - u(y', t')|}{[|t - t'|^{1/2} + d(y, y')]^\alpha} \leq \frac{A_\kappa}{r^\alpha} \sup_Q \{|u|\}$$

and conclusion follows, where $A_\kappa := 4\theta^{-1}(1 - \kappa)^{-\alpha}$. The second case is trivial. Indeed if $\rho > (1 - \kappa)r$, then the above inequality obviously holds. Therefore we complete the proof of theorem. \square

Using the Harnack inequality and the above Hölder continuity, we immediately derive following Liouville theorem.

Corollary 3.2. *Let $(M, g, e^{-f} dv)$ be an n -dimensional complete noncompact smooth metric measure space. Assume that $\text{Ric}_f \geq 0$ and $|f|(x) \leq A$ for some nonnegative constant A . Then M has the strong Liouville property. Moreover, there exists an $\alpha \in (0, 1]$ such that any f -harmonic function u which satisfies*

$$\lim_{r \rightarrow \infty} \left(r^{-\alpha} \cdot \sup_{B(o, r)} \{|u|\} \right) = 0$$

for some fixed $o \in M$ is constant.

Here, we say that a smooth metric measure space has the *strong Liouville property* if any solution u of the f -harmonic equation $\Delta_f u = 0$ on $(M, g, e^{-f} dv)$ which is bounded below (or above) is constant.

Remark 3.3. Corollary 3.2 was also proved by Munteanu and Wang [27]. We emphasize that this result can be regarded as a direct consequence of Theorem 1.1. If f is constant and $\alpha = 1$, then Corollary 3.2 returns to Cheng's Liouville property in [10]. If f is constant, this case appeared in [34] (see also Theorem 4.3 in [32]).

Proof of Corollary 3.2. We first prove the first part of theorem. The conditions of theorem make the parabolic Harnack inequality (Corollary 2.7, $K = 0$) hold and hence the corresponding elliptic Harnack inequality is also true. Assume that u is a solution of the f -harmonic equation which is bounded below. Let

$$m(u) := \inf_M \{u\}.$$

Applying the elliptic Harnack inequality in the ball $2B = B_o(2r)$ and to the non-negative function $v = u - m(u)$, we have that

$$\sup_B \{u - m(u)\} \leq C(n, A) \cdot \inf_B \{u - m(u)\}.$$

We let the radius of $B = B_o(r)$ tend to infinity, and obviously $\inf_B \{u - m(u)\}$ tends to zero. Therefore we conclude that $u = m(u)$ is constant.

In the following we will prove the second part of theorem. Let α be as given by Theorem 3.1. Let u be a function satisfying $\Delta_f u = 0$ and

$$\lim_{r \rightarrow \infty} \left(r^{-\alpha} \cdot \sup_{B_o(r)} \{|u|\} \right) = 0.$$

Note that in this case we can choose $\alpha = 1$. Because from the proof of Theorem 3.2, it turns out that if θ is sufficient small, then $\alpha \rightarrow 1$. Fix some $x \in M$ and y

such that $d(x, y) \leq 1$. Applying Theorem 3.1 to u in a ball $B_R = B_o(R)$ with R so large that $x, y \in \frac{1}{2}B_R$, we find that

$$|u(x) - u(y)| \leq \frac{C}{R^\alpha} \sup_{B_R} \{|u|\}.$$

Since the above inequality holds for all R large enough, we can let R tend to infinity to obtain that $|u(x) - u(y)| = 0$. Since $x, y \in M$ with $d(x, y) \leq 1$ are arbitrary and M is connected, we conclude that u must be constant. \square

4. TWO-SIDED GAUSSIAN BOUNDS ON f -HEAT KERNEL

In this section, we shall obtain upper and lower bound estimates for the f -heat kernel $H(x, y, t)$ on complete noncompact metric measure space. The proof seems to be different from the classical discussion of Li and Yau in [21]. Our argument is similar to the discussion in A. Grigor'yan [14] and L. Saloff-Coste [32].

First, we show that the Neumann Poincaré inequality and the volume doubling property imply a lower bound on the f -heat kernel. To achieve this, we start it by the following important lemma.

Lemma 4.1. *Under the same assumptions of Theorem 2.6, then there exist a constant $c_4 := c_4(n, A)$ such that, for any $x, y \in B_o(\frac{1}{2}r)$, and any $0 < s < t < \infty$ and any non-negative solution of the f -heat equation in $M \times (0, \infty)$,*

$$\ln \left(\frac{u(x, s)}{u(y, t)} \right) \leq c_4 \left[\left(K + \frac{1}{r^2} + \frac{1}{s} \right) (t - s) + \frac{d^2(x, y)}{t - s} \right].$$

Sketch proof of Lemma 4.1. Since Theorem 2.6 implies a parabolic Harnack inequality of the f -heat equation, it is sufficient to prove the above inequalities by carefully choosing different space-time solutions. Please see Corollary 5.4.4 in [34] or Corollary 5.4 in [33] for a detailed proof \square

Using Lemma 4.1, we can get a lower bound on the f -heat kernel on complete metric measure spaces.

Proposition 4.2. *Under the same assumptions of Theorem 2.6, then there exists a constant $c_5 := c_5(n, A)$ such that, for any $x, y \in B_o(\frac{1}{2}r)$ and any $0 < t < \infty$, the f -heat kernel $H(x, y, t)$ satisfies*

$$(4.1) \quad H(x, y, t) \geq H(x, x, t) \exp \left[-c_5 \left(1 + \frac{t}{r^2} + Kt + \frac{d^2(x, y)}{t} \right) \right].$$

Moreover, there exists a constant $c_6 := c_6(n, A)$ such that, for any $x, y \in B_o(\frac{1}{2}r)$ and any $0 < t < r^2 < R^2$

$$(4.2) \quad H(x, y, t) \geq \frac{e^{-c_6(1+Kt)}}{V_f(B_x(\sqrt{t}))} \exp \left(-c_5 \frac{d^2(x, y)}{t} \right).$$

Proof. The proof of this result follows from that of Theorem 5.4.11 in [34] with no much modification. In fact using Lemma 4.1, we let $u(y, t) = H(x, y, t)$ with x fixed and $s = t/2$ and then get (4.1), where we used the fact that $H(x, x, t)$ is non-increasing.

Then we prove (4.2). Note that the conditions of this proposition imply a parabolic Harnack inequality, which leads to the on-diagonal f -heat kernel lower bound

$$(4.3) \quad H(x, x, t) \geq e^{-c(n, A)(1+Kt)} \cdot V_f^{-1}(B_x(\sqrt{t}))$$

for all $x \in M$ and $0 < t < r^2 < R^2$. Indeed we fix $0 < t < r^2 < R$ and consider ϕ be a smooth function such that $0 \leq \phi \leq 1$, $\phi = 1$ on $B := B_x(\sqrt{t})$ and $\phi = 0$ on $M \setminus 2B$. Define

$$u(y, t) = \begin{cases} P_t \phi(y) & \text{if } t > 0 \\ \phi(y) & \text{if } t \leq 0, \end{cases}$$

$P_t = e^{t\Delta_f}$ be the heat semigroup of Δ_f on $L^2(M, \mu)$. Obviously, $u(y, t)$ satisfies $(\partial_t - \Delta_f)u = 0$ on $B \times (-\infty, \infty)$. Using the parabolic Harnack inequality, first to u , and then to the f -heat kernel $(y, s) \rightarrow H(x, y, s)$, we have

$$\begin{aligned} 1 = u(x, 0) &\leq e^{c(1+Kt)} u(x, t/2) \\ &= e^{c(1+Kt)} \int_{B(x, \sqrt{t})} H(x, y, t/2) \phi(y) d\mu(y) \\ &\leq e^{c(1+Kt)} \int_{B(x, 2\sqrt{t})} H(x, y, t/2) d\mu(y) \\ &\leq e^{2c(1+Kt)} V_f(B_x(2\sqrt{t})) H(x, x, t) \\ &\leq e^{2c(1+Kt)} V_f(B_x(\sqrt{t})) e^{2^{n+4A} e^{C(n,A)\sqrt{Kt}}} H(x, x, t), \end{aligned}$$

where in the last inequality we used (2.2). This gives (4.3) as desired. Hence (4.2) then easily follows by (4.1) and (4.3). \square

Secondly, we can show that the Neumann Poincaré inequality and the volume doubling property also imply an upper bound on the f -heat kernel. To achieve this, the following integral estimate is critical useful due to E. B. Davies [11].

Lemma 4.3 (E. B. Davies [11]). *Let $(M, g, e^{-f} dv)$ be an n -dimensional complete smooth metric measure space. Let $\lambda_1 > 0$ be the bottom of the L^2 -spectrum of the f -Laplacian. Assume that B_1 and B_2 are bounded subsets of M . Then*

$$\int_{B_1} \int_{B_2} H(x, y, t) d\mu(y) d\mu(x) \leq e^{-\lambda_1 t} V_f(B_1)^{1/2} V_f(B_2)^{1/2} \exp\left(-\frac{d^2(B_1, B_2)}{4t}\right),$$

where $d(B_1, B_2)$ denotes the distance between the sets B_1 and B_2 .

We are now ready to give an upper bound on the fundamental solution of the f -heat equation.

Proposition 4.4. *Under the same assumptions of Theorem 2.6, then there exist constants c_7 and c_8 such that, for any $x, y \in B_o(\frac{1}{2}r)$ and $0 < t < r^2/4$, the f -heat kernel $H(x, y, t)$ satisfies*

$$(4.4) \quad H(x, y, t) \leq \frac{e^{c_8(1+Kt)}}{V_f(B_x(\sqrt{t}))} \exp\left(-c_7 \frac{d^2(x, y)}{t}\right).$$

Proof. Fix a fixed $y \in B_o(r)$ and $\delta > 0$, applying Lemma 4.1 to the position solution $u(x, t) = H(x, y, t)$ by taking $s = t$ and $t = (1 + \delta)t$,

$$H(x, y, t) \leq H(x', y, (1 + \delta)t) \cdot \exp\left\{c_4 \left[\left(K + \frac{1}{r^2} + \frac{1}{t}\right) \delta t + \frac{d^2(x, x')}{\delta t}\right]\right\}.$$

Integrating over $x' \in B_x(\sqrt{t})$ gives

$$(4.5) \quad \begin{aligned} H(x, y, t) &\leq \exp \left[c_4 \left((K + r^{-2})\delta t + \delta + \frac{1}{\delta} \right) \right] V_f^{-1}(B_x(\sqrt{t})) \\ &\quad \times \int_{B_x(\sqrt{t})} H(x', y, (1 + \delta)t) d\mu(x'). \end{aligned}$$

Applying Lemma 4.1 and the same argument to the position solution

$$u(y, t) = \int_{B_x(\sqrt{t})} H(x', y, t) d\mu(x'),$$

by taking $s = (1 + \delta)t$ and $t = (1 + 2\delta)t$, we obtain

$$\begin{aligned} \int_{B_x(\sqrt{t})} H(x', y, (1 + \delta)t) d\mu(x') &\leq \exp \left[c_4 \left((K + r^{-2})\delta t + \delta + \frac{1}{\delta} \right) \right] V_f^{-1}(B_y(\sqrt{t})) \\ &\quad \times \int_{B_y(\sqrt{t})} \int_{B_x(\sqrt{t})} H(x', y', (1 + 2\delta)t) d\mu(x') d\mu(y'). \end{aligned}$$

Substituting this into (4.5) yields

$$\begin{aligned} H(x, y, t) &\leq \exp \left[2c_4 \left((K + r^{-2})\delta t + \delta + \frac{1}{\delta} \right) \right] V_f^{-1}(B_x(\sqrt{t})) V_f^{-1}(B_y(\sqrt{t})) \\ &\quad \times \int_{B_y(\sqrt{t})} \int_{B_x(\sqrt{t})} H(x', y', (1 + 2\delta)t) d\mu(x') d\mu(y'). \end{aligned}$$

Combining this with Lemma 4.3, we have

$$(4.6) \quad \begin{aligned} H(x, y, t) &\leq \exp \left[2c_4 \left((K + r^{-2})\delta t + \delta + \frac{1}{\delta} \right) - \lambda_1 t \right] \\ &\quad \times V_f^{-1/2}(B_x(\sqrt{t})) V_f^{-1/2}(B_y(\sqrt{t})) \exp \left[-\frac{d^2(B_x(\sqrt{t}), B_y(\sqrt{t}))}{4(1 + 2\delta)t} \right]. \end{aligned}$$

Notice that if $d(x, y) \leq 2\sqrt{t}$, then $d(B_x(\sqrt{t}), B_y(\sqrt{t})) = 0$ and hence

$$-\frac{d^2(B_x(\sqrt{t}), B_y(\sqrt{t}))}{4(1 + 2\delta)t} = 0 \leq 1 - \frac{d^2(x, y)}{4(1 + 2\delta)t},$$

and if $d(x, y) > 2\sqrt{t}$, then $d(B_x(\sqrt{t}), B_y(\sqrt{t})) = d(x, y) - 2\sqrt{t}$ hence

$$-\frac{d^2(B_x(\sqrt{t}), B_y(\sqrt{t}))}{4(1 + 2\delta)t} = -\frac{(d(x, y) - 2\sqrt{t})^2}{4(1 + 2\delta)t} \leq -\frac{d^2(x, y)}{4(1 + 2\delta)t} + \frac{1}{2\delta}.$$

Therefore in any case, (4.6) becomes

$$(4.7) \quad \begin{aligned} H(x, y, t) &\leq \exp \left[2c_4 \left((K + r^{-2})\delta t + \delta + \frac{1}{\delta} \right) - \lambda_1 t \right] \\ &\quad \times V_f^{-1/2}(B_x(\sqrt{t})) V_f^{-1/2}(B_y(\sqrt{t})) \exp \left(-\frac{d^2(x, y)}{4(1 + 2\delta)t} \right). \end{aligned}$$

Now we want to estimate $(K + r^{-2})\delta t + \delta + \frac{1}{\delta}$ in (4.7). Let

$$\delta = \min \left\{ \epsilon, [(K + r^{-2})t]^{-1/2} \right\}.$$

If $[(K + r^{-2})t]^{-1/2} \leq \epsilon$, then

$$(K + r^{-2})\delta t + \delta + \frac{1}{\delta} \leq 2[(K + r^{-2})t]^{1/2} + \epsilon.$$

If $[(K + r^{-2})t]^{-1/2} > \epsilon$, then we have

$$\begin{aligned} (K + r^{-2})\delta t + \delta + \frac{1}{\delta} &\leq [(K + r^{-2})t] \epsilon + \epsilon + \frac{1}{\epsilon} \\ &\leq [(K + r^{-2})t]^{1/2} + \epsilon + \frac{1}{\epsilon}. \end{aligned}$$

Hence, in either case, the right hand side of (4.7) can be estimate by

$$\begin{aligned} (4.8) \quad H(x, y, t) &\leq \exp \left[2c_4 \left(2[(K + r^{-2})t]^{1/2} + \epsilon + \frac{1}{\epsilon} \right) - \lambda_1 t \right] \\ &\quad \times V_f^{-1/2}(B_x(\sqrt{t})) V_f^{-1/2}(B_y(\sqrt{t})) \exp \left(-\frac{d^2(x, y)}{4(1 + 2\epsilon)t} \right). \end{aligned}$$

Moreover the volume doubling property implies (see, e.g., Lemma 5.2.7 in [34]) that

$$\begin{aligned} V_f(x, \sqrt{t}) &\leq C(n, A) \exp \left(C(n, A) \sqrt{Kt} \cdot \frac{d(x, y)}{\sqrt{t}} \right) V_f(y, \sqrt{t}) \\ &\leq C(n, A) \exp \left(\bar{C}(n, A, \epsilon) Kt + \frac{d^2(x, y)}{8(1 + 2\epsilon)t} \right) V_f(y, \sqrt{t}). \end{aligned}$$

Substituting this into (4.8) and using $0 < t < r^2/4$, then the theorem follows. \square

Combining Lemmas 2.1, 2.2 and Propositions 4.2, 4.4, we immediately obtain two-sided f -heat kernel bounds on smooth metric measure spaces.

Theorem 4.5. *Let $(M, g, e^{-f} dv)$ be an n -dimensional complete noncompact smooth metric measure space. If $\text{Ric}_f \geq -(n-1)K$ and $|f|(x) \leq A$ on $B_o(2r)$ for some nonnegative constants K and A , then there exist positive constants c_i , $i = 5, 6, 7, 8$, depending on n and A such that the f -heat kernel $H(x, y, t)$ satisfies*

$$\frac{e^{-c_6(1+Kt)}}{V_f(x, \sqrt{t})} \exp \left(-c_5 \frac{d^2(x, y)}{t} \right) \leq H(x, y, t) \leq \frac{e^{c_8(1+Kt)}}{V_f(x, \sqrt{t})} \exp \left(-c_7 \frac{d^2(x, y)}{t} \right)$$

for any $x, y \in B_o(r/2)$ and $0 < t < r^2/4$.

5. MEAN VALUE INEQUALITY

In this section, the main objective is to derive a mean value inequality on complete noncompact metric measure space, which is a natural generalization of the Li-Schoen's result in [20]. First, we give the following Poincaré inequality.

Theorem 5.1. *Let $(M, g, e^{-f} dv)$ be a complete noncompact smooth metric measure space. If $\text{Ric}_f \geq -(n-1)K$ and $|f|(x) \leq A$ for some nonnegative constants K and A , then for any $\alpha \geq 1$, there exists constants C_3 and C_4 depending only on α , n and A such that*

$$\int_{B_o(R)} |\phi|^\alpha d\mu \leq C_3 \left(\frac{R}{1 + \sqrt{KR}} \right)^\alpha e^{C_4(1 + \sqrt{KR})} \int_{B_o(R)} |\nabla \phi|^\alpha d\mu$$

for any compactly supported function ϕ on $B_o(R)$. In particular, the first Dirichlet eigenvalue μ_1 of f -Laplacian on $B_o(R)$ satisfies

$$\mu_1 \geq C_3^{-1} \left(\frac{R}{1 + \sqrt{KR}} \right)^{-2} e^{-C_4(1 + \sqrt{KR})}.$$

Sketch proof of Lemma 5.1. The proof is exactly the same as that of Corollary 1.1 proved by Li-Schoen [20] except that the classical Laplacian comparison is replaced by the generalized Laplacian comparison (see Lemma 2.1)

$$\begin{aligned}\Delta_f r(x) &\leq (n-1+4A)\sqrt{K} \coth \sqrt{K}r \\ &\leq \frac{n-1+4A}{r} + (n-1+4A)\sqrt{K}.\end{aligned}$$

Besides this, all the integration calculations should be done with respect to the new measure μ . To save the length of paper, we omit details of the proof. \square

We now proceed to derive the L^2 mean value inequality with respect to the weighted measure by Theorem 5.1.

Theorem 5.2. *Let $(M, g, e^{-f} dv)$ be a complete noncompact smooth metric measure space. Assume that $\text{Ric}_f \geq -(n-1)K$ with $|f|(x) \leq A$ for some nonnegative constants K and A . Let u be a nonnegative f -subharmonic function defined on $B_o(R)$. There exist a constant C_5 , depending only on n and A such that for any $\tau \in (0, 1/2)$*

$$\sup_{B_o((1-\tau)R)} u^2 \leq \tau^{-C_5(1+\sqrt{K}R)} V_f^{-1}(B_o(R)) \int_{B_o(R)} u^2 d\mu.$$

Proof. Let h be a harmonic function on $B_o((1-2^{-1}\tau)R)$ obtained by the solving the Dirichlet boundary problem

$$\Delta_f h = 0 \quad \text{on} \quad B_o((1-\tau/2)R),$$

and

$$h = u \quad \text{on} \quad \partial B_o((1-\tau/2)R).$$

Since u is nonnegative, by the maximum principle, function h is positive on the ball $B_o((1-2^{-1}\tau)R)$. Moreover,

$$u \leq h \quad \text{on} \quad B_o((1-\tau/2)R).$$

Using Lemmas 2.1, 2.2 and 2.4, by the Moser iteration argument as in [27], we have the following elliptic Harnack inequality

$$\sup_{B_o((1-\tau)R)} h \leq e^{c(n,A)(1+\sqrt{K}R)} \inf_{B_o((1-\tau)R)} h,$$

where c depends only on n and A . In particular,

$$\begin{aligned}(5.1) \quad \sup_{B_o((1-\tau)R)} u^2 &\leq \sup_{B_o((1-\tau)R)} h^2 \\ &\leq e^{c(n,A)(1+\sqrt{K}R)} \inf_{B_o((1-\tau)R)} h^2 \\ &\leq e^{c(n,A)(1+\sqrt{K}R)} V_f^{-1}(B_o((1-\tau)R)) \int_{B_o((1-\tau)R)} h^2 d\mu.\end{aligned}$$

Below we will estimate the $L^2(\mu)$ -norm of h in terms of the $L^2(\mu)$ -norm of u . By the triangle inequality, we have

$$\begin{aligned}(5.2) \quad \int_{B_o((1-\tau)R)} h^2 d\mu &\leq 2 \int_{B_o((1-\tau)R)} (h-u)^2 d\mu + 2 \int_{B_o((1-\tau)R)} u^2 d\mu \\ &\leq 2 \int_{B_o((1-\tau/2)R)} (h-u)^2 d\mu + 2 \int_{B_o(R)} u^2 d\mu.\end{aligned}$$

Since $(h - u)$ vanishes on $\partial B_o((1 - \tau/2)R)$ we can apply Theorem 5.1 to show that

$$\begin{aligned} \int_{B_o((1-\tau/2)R)} (h - u)^2 d\mu &\leq \frac{C_3 R^2}{(1 + \sqrt{K}R)^2} e^{C_4(1+\sqrt{K}R)} \int_{B_o((1-\tau/2)R)} |\nabla(h - u)|^2 d\mu \\ &\leq \frac{C_3 R^2 e^{C_4(1+\sqrt{K}R)}}{(1 + \sqrt{K}R)^2} \int_{B_o((1-\tau/2)R)} 2(|\nabla h|^2 + |\nabla u|^2) d\mu, \end{aligned}$$

where we have used the triangle inequality again. Since the Dirichlet integral of h is least among all functions which coincide with h on the boundary, from above we conclude that

$$(5.3) \quad \int_{B_o((1-\tau/2)R)} (h - u)^2 d\mu \leq \frac{4C_3 R^2 e^{C_4(1+\sqrt{K}R)}}{(1 + \sqrt{K}R)^2} \int_{B_o((1-\tau/2)R)} |\nabla u|^2 d\mu.$$

Now we use the fact that u is f -subharmonic to estimate the Dirichlet integral of u in terms of the L^2 -norm of u . We have for any ϕ with compact support in $B_o(R)$

$$\begin{aligned} 0 &\leq \int_{B_o(R)} \phi^2 u \Delta_f u d\mu \\ &= - \int_{B_o(R)} \phi^2 |\nabla u|^2 d\mu - 2 \int_{B_o(R)} \phi u \langle \nabla \phi, \nabla u \rangle d\mu \\ &\leq - \int_{B_o(R)} \phi^2 |\nabla u|^2 d\mu + 2 \left(\int_{B_o(R)} \phi^2 |\nabla u|^2 d\mu \right)^{1/2} \left(\int_{B_o(R)} u^2 |\nabla \phi|^2 d\mu \right)^{1/2}. \end{aligned}$$

Thus

$$\int_{B_o(R)} \phi^2 |\nabla u|^2 d\mu \leq 4 \int_{B_o(R)} u^2 |\nabla \phi|^2 d\mu.$$

We let $\phi(r(x))$ be a cut-off function given by a function of $r(x) = r(o, x)$ alone, such that $\phi(r) = 1$ on $B_o((1 - \tau/2)R)$, $\phi(r) = 0$ on $\partial B_o(R)$, and satisfying

$$|\nabla \phi| \leq \frac{c}{\tau R}.$$

Then the above inequality becomes

$$\int_{B_o((1-\tau/2)R)} |\nabla u|^2 d\mu \leq \frac{4c^2}{\tau^2 R^2} \int_{B_o(R)} u^2 d\mu.$$

Combining this with (5.1), (5.2) and (5.3) yields

$$\begin{aligned} (5.4) \quad \sup_{B_o((1-\tau)R)} u^2 &\leq C \left(\frac{32c^2 C_3 \tau^{-2} e^{C_4(1+\sqrt{K}R)}}{(1 + \sqrt{K}R)^2} + 2 \right) V_f^{-1}(B_o((1 - \tau)R)) \int_{B_o(R)} u^2 d\mu \\ &\leq C_6 \tau^{-C_7(1+\sqrt{K}R)} e^{C_8(1+\sqrt{K}R)} V_f^{-1}(B_o((1 - \tau)R)) \int_{B_o(R)} u^2 d\mu \end{aligned}$$

for some new constants $C_i = C_i(n, A)$, $i = 6, 7, 8$. To finish the proof, we also need to estimate the f -volume of $B_o(R)$ in terms of the volume of $B_o((1 - \tau)R)$. Recall the bound for $\Delta_f r^2$:

$$\Delta_f r^2 \leq 2(n + 4A) + 2\sqrt{K}(n - 1 + 4A)r,$$

and hence

$$\int_{B_o(t)} \Delta_f r^2 d\mu \leq 2(n+4A)V_f(t) + 2\sqrt{K}(n-1+4A) \int_{B_o(t)} r d\mu,$$

where $V_f(t) = \text{Vol}_f(B_o(t))$. By Green formula, since

$$\int_{B_o(t)} \Delta_f r^2 d\mu = \int_{\partial B_o(t)} \frac{\partial r^2}{\partial r} d\sigma = 2t \frac{\partial V_f(B_o(t))}{\partial t},$$

then

$$tV_f'(t) \leq (n+4A)V_f(t) + \sqrt{K}(n-1+4A)tV_f(t).$$

Hence the function $t^{-(n+4A)}e^{-\sqrt{K}(n-1+4A)t}V_f(t)$ is decreasing in $t \geq 0$. Therefore

$$(5.5) \quad V_f^{-1}(B_o((1-\tau)R)) \leq V_f^{-1}(B_o(R)) \left(\frac{1}{1-\tau} \right)^{n+4A} \cdot e^{\sqrt{K}R\tau(n-1+4A)},$$

where $0 < \tau < 1/2$. Combining this with (5.4) completes the proof of theorem. \square

In the following, we show that the L^p mean value inequality for any $p \in (0, 2]$ is a formal consequence of that given in Theorem 5.2.

Theorem 5.3. *Under the same assumption of Theorem 5.2, for any $p \in (0, 2]$, there exists a constant c depending only on n , p , and A such that*

$$\sup_{B_o((1-\tau)R)} u^p \leq \tau^{-c(1+\sqrt{K}R)} V_f^{-1}(R) \int_{B_o(R)} u^p d\mu$$

for any $\tau \in (0, 1/2)$, where $V_f^{-1}(R) := V_f^{-1}(B_o(R))$.

Proof. By Theorem 5.2, for any $\delta \in (0, 1/2]$, $\theta \in [1/2, 1-\delta]$, we have

$$\sup_{B_o(\theta R)} u^2 \leq \delta^{-C_5(1+\sqrt{K}R)} V_f^{-1}((\theta+\delta)R) \int_{B_o((\theta+\delta)R)} u^2 d\mu.$$

Since $\theta + \delta \geq 1/2$, this inequality implies

$$\sup_{B_o(\theta R)} u^2 \leq \delta^{-C_5(1+\sqrt{K}R)} V_f^{-1}(2^{-1}R) \int_{B_o((\theta+\delta)R)} u^2 d\mu.$$

We also note that

$$\int_{B_o((\theta+\delta)R)} u^2 d\mu \leq \left(\sup_{B_o((\theta+\delta)R)} u^2 \right)^{1-p/2} \int_{B_o((\theta+\delta)R)} u^p d\mu.$$

Hence

$$\sup_{B_o(\theta R)} u^2 \leq \delta^{-C_5(1+\sqrt{K}R)} V_f^{-1}(2^{-1}R) \left(\sup_{B_o((\theta+\delta)R)} u^2 \right)^{1-p/2} \int_{B_o((\theta+\delta)R)} u^p d\mu.$$

If we set

$$M(\theta) := \sup_{B_o(\theta R)} u^2$$

and

$$N := V_f^{-1}(2^{-1}R) \int_{B_o(R)} u^p d\mu,$$

we have shown

$$M(\theta) \leq N \delta^{-C_5(1+\sqrt{K}R)} M(\theta+\delta)^{1-p/2}$$

for any $\delta \in (0, 1/2]$ and $\theta \in [1/2, 1 - \delta]$. Choosing

$$\theta_0 = 1 - \tau \quad \text{and} \quad \theta_i = \theta_{i-1} + 2^{-i}\tau$$

for $i = 1, 2, 3, \dots$, we have that

$$M(\theta_{i-1}) \leq N_1 2^{iC_5(1+\sqrt{KR})} M(\theta_i)^\lambda,$$

where $\lambda = 1 - p/2$ and $N_1 = N\tau^{-C_5(1+\sqrt{KR})}$. Iterating yields

$$M(\theta_0) \leq K_1^{\sum_{i=1}^j \lambda^{i-1}} 2^{C_5(1+\sqrt{KR}) \sum_{i=1}^j i \lambda^{i-1}} M(\theta_j)^{\lambda^j}$$

for any $j \geq 1$. Letting j tend to infinity yields

$$M(\theta_0) \leq \tau^{-C_9(1+\sqrt{KR})} \left[V_f^{-1}(2^{-1}R) \int_{B_o(R)} u^p d\mu \right]^{2/p},$$

where C_9 depends only on n, p and A . By the definition of $M(\theta_0)$, we have

$$\sup_{B_o((1-\tau)R)} u^p \leq \tau^{-2^{-1}pC_9(1+\sqrt{KR})} V_f^{-1}(2^{-1}R) \int_{B_o(R)} u^p d\mu.$$

Finally, by the relation (5.5), i.e.,

$$V_f^{-1}(2^{-1}R) \leq C(n, A) e^{C(1+\sqrt{KR})} V_f^{-1}(R),$$

the theorem follows. \square

6. L^p -LIOUVILLE THEOREM

In this section, we will study various L^p Liouville theorems on complete noncompact smooth metric measure spaces. In the course of proving our results, integrations by parts, cut-off function techniques and comparison theorems are often used, which are more or less similar to classical results with respect to Ricci curvature case in [18], [20] and [43].

6.1. The $1 < p < \infty$ case. The proof of this case is nearly the same as the classical case for the Ricci curvature proved by Yau [43] with little modification, but is included for completeness.

Theorem 6.1. *Let $(M, g, e^{-f} dv)$ be an n -dimensional complete smooth metric measure space. Suppose $o \in M$ is a fixed point such that the geodesic ball $B_o(2R)$ centered at o of radius $2R$ satisfies $B_o(2R) \cap \partial M = \emptyset$. Let u be a nonnegative f -subharmonic function defined on $B_o(2R)$. Then for any constant $1 < p < \infty$, there exists a constant $C(p) > 0$ depending only on p such that*

$$\int_{B_o(R)} u^{p-2} |\nabla u|^2 d\mu \leq \frac{C}{R^2} \int_{B_o(2R)} u^p d\mu.$$

In particular, if M has no boundary, then there does not exist any nonconstant, nonnegative, $L^p(\mu)$ -integrable f -subharmonic function. This constant must be zero if M has infinite f -volume.

Proof of Theorem 6.1. The proof is very similar to the Yau's arguments [43]. Let $\phi(r(x))$ be a cut-off function given by a function of $r(x) = r(o, x)$ alone, whose definition is as follows:

$$\phi(r) = \begin{cases} 1 & \text{if } r \leq R, \\ 0 & \text{if } r \geq 2R, \end{cases}$$

with $\phi(r) \geq 0$, and $0 \leq (\phi')^2/\phi \leq CR^{-2}$ for some positive constant C . Obviously, we have

$$\begin{aligned} 0 &\leq \int_{B_o(2R)} \phi^2 u^{p-1} \Delta_f u d\mu \\ &= -(p-1) \int_{B_o(2R)} \phi^2 u^{p-2} |\nabla u|^2 d\mu - 2 \int_{B_o(2R)} \phi u^{p-1} \langle \nabla \phi, \nabla u \rangle d\mu. \end{aligned}$$

Note that

$$2 \int_{B_o(2R)} \phi u^{p-1} \langle \nabla \phi, \nabla u \rangle \leq \frac{p-1}{2} \int_{B_o(2R)} \phi^2 u^{p-2} |\nabla u|^2 + \frac{2}{p-1} \int_{B_o(2R)} u^p |\nabla \phi|^2.$$

Hence we have

$$\begin{aligned} \frac{k-1}{2} \int_{B_p(R)} u^{p-2} |\nabla u|^2 d\mu &\leq \frac{p-1}{2} \int_{B_o(2R)} \phi^2 u^{p-2} |\nabla u|^2 d\mu \\ &\leq \frac{2}{p-1} \int_{B_o(2R)} u^p |\nabla \phi|^2 d\mu \\ &\leq \frac{C}{R^2(p-1)} \int_{B_o(2R)} u^p d\mu. \end{aligned}$$

If M has no boundary, by taking $R \rightarrow \infty$, the fact that u is $L^p(\mu)$ implies

$$\int_{B_o(R)} u^{p-2} |\nabla u|^2 d\mu = 0,$$

which furthermore implies $|\nabla u| = 0$. Hence u must be constant. \square

6.2. The $0 < p < 1$ case. For $0 < p < 1$, following the work of Li-Schoen [20], we have the following result.

Theorem 6.2. *Let $(M, g, e^{-f} dv)$ be an n -dimensional complete noncompact smooth metric measure space. If*

$$Ric_f \geq -\delta(n)(1+r(x))^{-2},$$

where $r(x)$ is the distance to a fixed point $o \in M$, and f is bounded, then any nonnegative L^p -integrable ($0 < p < 1$) f -subharmonic function must be identically constant. Moreover, this constant must be zero if M has infinite f -volume.

Proof of Theorem 6.2. Since the arguments leading to Theorem 5.3 are local, by choosing more or less $\tau = 1/2$, we have the following L^p mean value inequality

$$(6.1) \quad \sup_{B_o(R/2)} u^p \leq \cdot 2^{c(1+\sqrt{K(x,5R)R})} V_f^{-1}(B_o(R)) \int_{B_o(R)} u^p d\mu$$

for nonnegative f -subharmonic functions u on $B_x(5R)$, where $Ric_f \geq -(n-1)K(x, 5R)$ with $|f|(x) \leq A$ for some nonnegative constants K and A on $B_x(5R)$. Here constant c depends on n , p and A . In the following, we will utilize (6.1) to show that u must vanish at infinity if nonnegative function u is f -subharmonic on M with $u \in L^p(\mu)$ ($0 < p < 1$). In fact, by the volume comparison theorem mentioned above, under the hypothesis on Ric_f and f , M must be of f -infinite volume and u must be identically zero.

Let $x \in M$ and consider a minimal geodesic γ joining o to x such that $\gamma(0) = o$ and $\gamma(T) = x$, where $T = r(o, x)$. We then define a set of values $\{t_i \in [0, T]\}_{i=0}^k$ satisfying

$$t_0 = 0, \quad t_1 = 1 + \beta, \quad \dots, \quad t_i = 2 \sum_{j=0}^i \beta^j - 1 - \beta^i,$$

where $\beta > 1$ to be chosen later, and $t_k = 2 \sum_{j=0}^k \beta^j - 1 - \beta^k$ is the largest such value with $t_k < T$. We denote the points $x_i = \gamma(t_i)$ and they obviously satisfy

$$r(x_i, x_{i+1}) = \beta^i + \beta^{i+1}, \quad r(o, x_i) = t_i \quad \text{and} \quad r(x_k, x) < \beta^k + \beta^{k+1}.$$

Moreover, the set of geodesic balls $B_{x_i}(\beta^i)$ cover $\gamma([0, 2 \sum_{j=0}^k \beta^j - 1])$ and they have disjoint interiors. We now claim that

$$(6.2) \quad V_f(B_{x_k}(\beta^k)) \geq C \left(\frac{\beta^{n+4A}}{(\beta+2)^{n+4A} - \beta^{n+4A}} \right)^k V_f(B_o(1))$$

for a fixed

$$\beta > \frac{2}{2^{1/n} - 1} > 1.$$

The proof of this claim essentially follows the arguments of Li-Schoen [20]. For the sake of completeness, we will outline the proof of this claim again.

For each $1 \leq i \leq k$, a relative comparison theorem (see (4.10) in [39]) argument shows that

$$\begin{aligned} V_f(B_{x_i}(\beta^i)) &\geq D_i [V_f(B_{x_i}(\beta^i + 2\beta^{i-1})) - V_f(B_{x_i}(\beta^i))] \\ &\geq D_i V_f(B_{x_{i-1}}(\beta^{i-1})), \end{aligned}$$

where

$$D_i = \frac{\int_0^{\beta^i \sqrt{K(x_i, \beta^i + 2\beta^{i-1})}} \sinh^{n-1+4A} t dt}{\int_{\beta^i \sqrt{K(x_i, \beta^i + 2\beta^{i-1})}}^{(\beta^i + 2\beta^{i-1}) \sqrt{K(x_i, \beta^i + 2\beta^{i-1})}} \sinh^{n-1+4A} t dt},$$

since $Ric_f \geq -(n-1)K(x_i, \beta^i + 2\beta^{i-1})$ and $|f|(x) \leq A$ for some nonnegative constants K and A on $B_{x_i}(\beta^i + 2\beta^{i-1})$. Iterating this inequality, we conclude that

$$(6.3) \quad V_f(B_{x_k}(\beta^k)) \geq V_f(B_o(1)) \prod_{i=1}^k D_i.$$

Since $r(o, x_i) = 2 \sum_{j=0}^i \beta^j - 1 - \beta^i$, the curvature assumption implies that

$$K(x_i, \beta^i + 2\beta^{i-1}) \leq \delta(n) \left(2 \sum_{j=0}^{i-2} \beta^j \right)^{-2}$$

for $i > 2$. Hence

$$\begin{aligned} \beta^i \sqrt{K(x_i, \beta^i + 2\beta^{i-1})} &\leq \frac{\sqrt{\delta(n)}}{2} \cdot \frac{\beta^i}{2 \sum_{j=0}^{i-2} \beta^j} \\ &\leq \frac{\sqrt{\delta(n)}}{2} \cdot \frac{\beta^2}{2 \sum_{j=0}^{i-2} \beta^{-j}} \\ &\leq \frac{\sqrt{\delta(n)}}{2} \cdot \beta(\beta-1)(1-\beta^{1-i})^{-1} \end{aligned}$$

which can be made arbitrarily small for a fixed β by choosing $\delta(n)$ to be sufficiently small. Hence D_i has the following approximation

$$\begin{aligned} D_i &\sim \frac{(\beta^i)^{n+4A}}{(\beta^i + 2\beta^{i-1})^{n+4A} - (\beta^i)^{n+4A}} \\ &= \frac{\beta^{n+4A}}{(\beta + 2)^{n+4A} - \beta^{n+4A}} \end{aligned}$$

by simply approximating $\sinh t$ with t . Hence (6.2) follows by combining (6.3).

In the following, we shall estimate $V_f(B_x(\beta^{k+1}))$. We achieve it by two cases.

Case 1: $r(x, x_k) \leq \beta^k(\beta - 1)$. In this case, we see that

$$B_{x_k}(\beta^k) \subset B_x(\beta^{k+1}),$$

and hence we know

$$V_f(B_{x_k}(\beta^k)) \leq V_f(B_x(\beta^{k+1})).$$

Combining this with (6.2), we conclude that

$$V_f(B_x(\beta^{k+1})) \geq C \left(\frac{\beta^{n+4A}}{(\beta + 2)^{n+4A} - \beta^{n+4A}} \right)^k V_f(B_o(1)).$$

Case 2: $r(x, x_k) > \beta^k(\beta - 1)$. In this setting, we see that

$$B_{x_k}(\beta^k) \subset B_x(r(x, x_k) + \beta^k) \setminus B_x(r(x, x_k) - \beta^k).$$

Using a relative comparison theorem, we have that

$$\begin{aligned} V_f(B_{x_k}(\beta^k)) &\geq D [V_f(B_x(r(x, x_k) + \beta^k)) - V_f(B_x(r(x, x_k) - \beta^k))] \\ &\geq D \cdot V_f(B_{x_k}(\beta^k)), \end{aligned}$$

where

$$D = \frac{\int_0^{\beta^k} \sqrt{K(x, r(x, x_k) + \beta^k)} \sinh^{n-1+4A} t dt}{\int_{(r(x, x_k) - \beta^k)}^{(r(x, x_k) + \beta^k)} \sqrt{K(x, r(x, x_k) + \beta^k)} \sinh^{n-1+4A} t dt}$$

Argument as above, since

$$\begin{aligned} (r(x, x_k) + \beta^k) \sqrt{K(x, r(x, x_k) + \beta^k)} &\leq (\beta^{k+1} + 2\beta^k) \sqrt{K(x, r(x, x_k) + \beta^k)} \\ &\leq \frac{\sqrt{\delta(n)}}{2} \cdot \beta(\beta - 1) \end{aligned}$$

can be made sufficiently small, we can approximate D by

$$D \sim \frac{\beta^{n+4A}}{(\beta + 2)^{n+4A}}.$$

Combining this with (6.2) yields

$$\begin{aligned} (6.4) \quad V_f(B_x(\beta^{k+1})) &\geq \frac{C\beta^{n+4A}}{(\beta + 2)^{n+4A}} \left(\frac{\beta^{n+4A}}{(\beta + 2)^{n+4A} - \beta^{n+4A}} \right)^{k+1} V_f(B_o(1)) \\ &\geq \tilde{C} \left(\frac{\beta^{n+4A}}{(\beta + 2)^{n+4A} - \beta^{n+4A}} \right)^k V_f(B_o(1)), \end{aligned}$$

where \tilde{C} depends on n , A and β . In any case, (6.4) is valid.

If we let $x \rightarrow \infty$, the value $k \rightarrow \infty$. Note that the choice of β ensures that

$$\frac{\beta^{n+4A}}{(\beta+2)^{n+4A} - \beta^{n+4A}} > 1,$$

and hence the right hand side of (6.4) tends to infinity. Now by (6.1) and the assumption of theorem, we have

$$u^p(x) \leq CV_f^{-1}(B_o(R)),$$

where C also depends on the L^p -norm of u . Using the value $R = \beta^{k+1}$ in the above inequality, the right hand side of the above inequality vanishes, thus proving that $u(x) \rightarrow 0$ as $x \rightarrow \infty$ and Theorem 6.2 follows by the maximum principle. \square

6.3. The $p = 1$ case. In the end of this section, we will consider the case $p = 1$. The proof of this case is a little complex. We shall follow the arguments of Li [18] (see also [19] and [22]) to derive the following result.

Theorem 6.3. *Let $(M, g, e^{-f} dv)$ be an n -dimensional complete noncompact smooth metric measure space. Assume that*

$$\text{Ric}_f \geq -C(1 + r^2(x))$$

for some constant $C > 0$, and $|f|(x) \leq A$ for some constant A . Then any nonnegative $L^1(\mu)$ -integrable f -subharmonic function must be identically constant. Moreover, this constant must be zero if M has infinite f -volume.

Following the trick of Li [18] (see also [22]) to prove Theorem 6.3, at first, we need the following integration by parts formula.

Theorem 6.4. *Let $(M, g, e^{-f} dv)$ be an n -dimensional complete noncompact smooth metric measure space. Assume that*

$$\text{Ric}_f \geq -C(1 + r^2(x))$$

for some constant $C > 0$, and $|f|(x) \leq A$ for some constant A . Then for any nonnegative $L^1(\mu)$ -integrable f -subharmonic function g ,

$$\int_M \Delta_f g H(x, y, t) d\mu(y) = \int_M H(x, y, t) \Delta_f g(y) d\mu(y).$$

Proof of Theorem 6.4. Similar to the arguments of [22], applying the Green formula on $B_o(R)$, we have

$$\begin{aligned} & \left| \int_{B_o(R)} \Delta_f g H(x, y, t) d\mu(y) - \int_{B_o(R)} H(x, y, t) \Delta_f g(y) d\mu(y) \right| \\ &= \left| \int_{\partial B_o(R)} \frac{\partial}{\partial r} H(x, y, t) g(y) d\mu_{\sigma, R}(y) - \int_{\partial B_o(R)} H(x, y, t) \frac{\partial}{\partial r} g(y) d\mu_{\sigma, R}(y) \right| \\ &\leq \int_{\partial B_o(R)} |\nabla H|(x, y, t) g(y) d\mu_{\sigma, R}(y) + \int_{\partial B_o(R)} H(x, y, t) |\nabla g|(y) d\mu_{\sigma, R}(y), \end{aligned}$$

where $\mu_{\sigma, R}$ denotes the weighted are-measure induced by μ on $\partial B_o(R)$. In the following we shall prove that the above two boundary integrals vanish as $R \rightarrow \infty$, which can be achieved by four steps.

Step 1. Under the curvature assumption of Theorem 6.4, by the mean value inequality (see Theorem 5.3), there exist a constant $c = c(n, A)$ such that

$$(6.5) \quad \sup_{B_o(R)} g(x) \leq e^{c(1+R\sqrt{K})} V_f^{-1}(2R) \int_{B_o(2R)} g(y) d\mu(y).$$

Consider $\phi(y) = \phi(r(y))$ to be a nonnegative cut-off function such that $0 \leq \phi \leq 1$, $|\nabla \phi| \leq \sqrt{3}$ and

$$\phi(x) = \begin{cases} 1 & \text{on } B_o(R+1) \setminus B_o(R), \\ 0 & \text{on } B_o(R-1) \cup (M \setminus B_o(R+2)). \end{cases}$$

Since g is f -subharmonic function, by the Schwarz inequality we have

$$\begin{aligned} 0 &\leq \int_M \phi^2 g \Delta_f g d\mu = - \int_M \nabla(\phi^2 g) \nabla g d\mu \\ &= -2 \int_M \phi g \langle \nabla \phi, \nabla g \rangle d\mu - \int_M \phi^2 |\nabla g|^2 d\mu \\ &\leq 2 \int_M |\nabla \phi|^2 g^2 d\mu - \frac{1}{2} \int_M \phi^2 |\nabla g|^2 d\mu. \end{aligned}$$

Then using the definition of ϕ and (6.5), we have that

$$\begin{aligned} \int_{B_o(R+1) \setminus B_o(R)} |\nabla g|^2 d\mu &\leq 4 \int_M |\nabla \phi|^2 g^2 d\mu \\ &\leq 12 \int_{B_o(R+2)} g^2 d\mu \\ &\leq 12 \sup_{B_o(R+2)} g \cdot \int_{B_o(R+2)} g d\mu \\ &\leq \frac{12e^{c(1+R\sqrt{K})}}{V_f(2R+4)} \cdot \left(\int_{B_o(R+2)} g d\mu \right)^2. \end{aligned}$$

From this, using the Schwarz inequality, we get

$$\begin{aligned} \int_{B_o(R+1) \setminus B_o(R)} |\nabla g| d\mu &\leq \left(\int_{B_o(R+1) \setminus B_o(R)} |\nabla g|^2 d\mu \right)^{1/2} \cdot [V_f(R+1) V_f(R)]^{1/2} \\ &\leq \left(\int_{B_o(R+1) \setminus B_o(R)} |\nabla g|^2 d\mu \right)^{1/2} \cdot V_f(2R+4)^{1/2}. \end{aligned}$$

Combining the above two inequalities, we have

$$(6.6) \quad \int_{B_o(R+1) \setminus B_o(R)} |\nabla g| d\mu \leq C_{10} e^{c(1+R\sqrt{K})} \int_{B_o(R+2)} g d\mu,$$

where $C_{10} = C_{10}(n, A)$.

Step 2. We estimate the f -heat kernel $H(x, y, t)$. By Theorem 4.5, the f -heat kernel $H(x, y, t)$ satisfies

$$(6.7) \quad H(x, y, t) \leq \frac{e^{c_8(1+Kt)}}{V_f(B_x(\sqrt{t}))} \exp \left(-c_7 \frac{d^2(x, y)}{t} \right)$$

for all $x, y \in B_o(R)$ and $0 < t < R^2/8$, where the constants C_7 and C_8 depending on n and A . Now combining (6.6) with (6.7) gives

$$\begin{aligned} J_1 &:= \int_{B_o(R+1) \setminus B_o(R)} H(x, y, t) |\nabla g|(y) d\mu(y) \\ &\leq \left(\sup_{y \in B_o(R+1) \setminus B_o(R)} H(x, y, t) \right) \int_{B_o(R+1) \setminus B_o(R)} |\nabla g| d\mu \\ &\leq \frac{C_{11} e^{cR\sqrt{K}}}{V_f(B_x(\sqrt{t}))} \cdot \exp \left[c_8 K t + \frac{-c_7(R-r)^2}{t} \right] \int_{B_o(R+2)} g d\mu \\ &= \frac{C_{11}}{V_f(B_x(\sqrt{t}))} \int_{B_o(R+2)} g \\ &\quad \times \exp \left[c\sqrt{K}R + c_8 K t - \frac{c_7 R^2}{t} + \frac{2c_7 R r}{t} - \frac{c_7 r^2}{t} \right], \end{aligned}$$

where $r = r(o, x)$ and $C_{11} = C_{11}(n, A)$. We now confirm that $J_1 \rightarrow 0$ as $R \rightarrow \infty$. Indeed, for any fixed point $x \in M$, if we choose time t sufficiently small fixed such that $-\frac{c_7}{t}$ dominates a major part with respect to the quadratic inequality in R , then J_1 tends to zero as R tends to infinity.

Step 3. Below we shall estimate the gradient of H . We consider the integral with respect to $d\mu$:

$$\begin{aligned} \int_M \phi^2(y) |\nabla H|^2(x, y, t) &= -2 \int_M \langle H(x, y, t) \nabla \phi(y), \phi(y) \nabla H(x, y, t) \rangle \\ &\quad - \int_M \phi^2(y) H(x, y, t) \Delta_f H(x, y, t) \\ &\leq 2 \int_M |\nabla \phi|^2(y) H^2(x, y, t) + \frac{1}{2} \int_M \phi^2(y) |\nabla H|^2(x, y, t) \\ &\quad - \int_M \phi^2(y) H(x, y, t) \Delta_f H(x, y, t). \end{aligned}$$

This implies

$$\begin{aligned} &\int_{B_o(R+1) \setminus B_o(R)} |\nabla H|^2 \\ &\leq \int_M \phi^2(y) |\nabla H|^2(x, y, t) \\ (6.8) \quad &\leq 4 \int_M |\nabla \phi|^2 H^2 - 2 \int_M \phi^2 H \Delta_f H \\ &\leq 12 \int_{B_o(R+1) \setminus B_o(R-1)} H^2 + 2 \int_{B_o(R+2) \setminus B_o(R-1)} H |\Delta_f H| \\ &\leq 12 \int_{B_o(R+2) \setminus B_o(R-1)} H^2 + 2 \left(\int_{B_o(R+2) \setminus B_o(R-1)} H^2 \right)^{\frac{1}{2}} \left(\int_M (\Delta_f H)^2 \right)^{\frac{1}{2}}. \end{aligned}$$

We point out that it is known that the heat semi-group is contractive in L^1 , hence

$$\int_M H(x, y, t) d\mu(y) \leq 1.$$

Using this and (6.7), we can estimate

$$(6.9) \quad \int_{B_o(R+2) \setminus B_o(R-1)} H^2(x, y, t) d\mu \leq \sup_{y \in B_o(R+2) \setminus B_o(R-1)} H(x, y, t) \\ \leq \frac{C_{12}}{V_f(B_x(\sqrt{t}))} \times \exp \left[c_8 K t + \frac{-c_7(R - r(o, x))^2}{t} \right].$$

Also, we *claim* that there exists a constant $C_{13} > 0$ such that

$$(6.10) \quad \int_M (\Delta_f H)^2(x, y, t) d\mu \leq \frac{C_{13}}{t^2} H(x, x, t).$$

To prove this inequality, we first derive the inequality for any Dirichlet f -heat kernel H defined on a compact subdomain of M . Using the fact that f -heat kernel on M can be obtained by taking limits of Dirichlet f -heat kernels on a compact exhaustion of M , then (6.10) follows. Indeed, if $H(x, y, t)$ is a Dirichlet f -heat kernel on a compact subdomain $\Omega \subset M$, using the eigenfunction expansion, then $H(x, y, t)$ can be written as the form

$$H(x, y, t) = \sum_i^\infty e^{-\lambda_i t} \psi_i(x) \psi_i(y),$$

where $\{\psi_i\}$ are orthonormal basis of the space of L^2 functions with Dirichlet boundary value satisfying the equation

$$\Delta_f \psi_i = -\lambda_i \psi_i.$$

Differentiating with respect to the variable y , we have

$$\Delta_f H(x, y, t) = - \sum_i^\infty \lambda_i e^{-\lambda_i t} \psi_i(x) \psi_i(y).$$

Noticing that $s^2 e^{-2s} \leq C_{13} e^{-s}$ for all $0 \leq s < \infty$, therefore

$$\int_M (\Delta_f H)^2 d\mu(y) \leq C_{13} t^{-2} \sum_i^\infty e^{-\lambda_i t} \psi_i^2(x) \\ = C_{13} t^{-2} H(x, x, t)$$

and claim (6.10) follows. Now combining (6.8), (6.9) and (6.10), we obtain

$$\int_{B_o(R+1) \setminus B_o(R)} |\nabla H|^2 d\mu \leq C_{14} \left[V_f^{-1} + t^{-1} V_f^{-\frac{1}{2}} H^{\frac{1}{2}}(x, x, t) \right] \\ \times \exp \left[c_8 K t + \frac{-c_7(R - r(o, x))^2}{t} \right],$$

where $V_f := V_f(B_x(\sqrt{t}))$. Applying Schwarz inequality and volume comparison theorem, we have

$$\begin{aligned}
 (6.11) \quad \int_{B_o(R+1) \setminus B_o(R)} |\nabla H| d\mu &\leq [V_f(B_o(R+1)) \setminus V_f(B_o(R))]^{1/2} \\
 &\quad \times \left[\int_{B_o(R+1) \setminus B_o(R)} |\nabla H|^2 d\mu \right]^{1/2} \\
 &\leq C(n, K, A) \left[V_f^{-1} + t^{-1} V_f^{-\frac{1}{2}} H^{\frac{1}{2}}(x, x, t) \right]^{1/2} \\
 &\quad \times \exp \left[\frac{c_8}{2} Kt + \frac{-c_7(R - d(o, x))^2}{2t} \right]
 \end{aligned}$$

for sufficiently large $R > 0$. On the other hand, by (6.5), (6.11) and Schwarz inequality we see that

$$\begin{aligned}
 J_2 &:= \int_{B_o(R+1) \setminus B_o(R)} |\nabla H(x, y, t)| g(y) d\mu(y) \\
 &\leq \sup_{y \in B_o(R+1) \setminus B_o(R)} g(y) \cdot \int_{B_o(R+1) \setminus B_o(R)} |\nabla H(x, y, t)| d\mu(y) \\
 &\leq \frac{C_{15} e^{c\sqrt{KR}}}{V_f(B_o(2R+2))} \int_{B_o(2R+2)} g(y) d\mu(y) \cdot \left[V_f^{-1} + t^{-1} V_f^{-\frac{1}{2}} H^{\frac{1}{2}}(x, x, t) \right]^{1/2} \\
 &\quad \times \exp \left[\frac{c_8}{2} Kt + \frac{-c_7(R - d(o, x))^2}{2t} \right].
 \end{aligned}$$

Similar to the discussion of J_1 , for any a fixed point $x \in M$, and any sufficiently small fixed $t > 0$, J_2 also tends to zero when R tends to infinity.

Step 4. Recall that the co-area formula states that for all $h \in C_0^\infty(M)$,

$$\int_{B_o(R+1) \setminus B_o(R)} h(y) d\mu(y) = \int_R^{R+1} \left[\int_{\partial B_o(r)} B_o(r) h(y) d\mu_{\sigma, r}(y) \right] dr.$$

By the mean value theorem, for any $R > 0$ there exists $\bar{R} \in (R, R+1)$ such that

$$\begin{aligned}
 J &:= \int_{\partial B_o(\bar{R})} [H(x, y, t) |\nabla g|(y) + |\nabla H|(x, y, t) g(y)] d\mu_{\sigma, \bar{R}}(y) \\
 &= \int_{B_o(R+1) \setminus B_o(R)} [H(x, y, t) |\nabla g|(y) + |\nabla H|(x, y, t) g(y)] d\mu(y) \\
 &= J_1 + J_2.
 \end{aligned}$$

By step 1 and step 2, we know that for any a fixed point $x \in M$, and for any sufficient small fixed t , J tends to zero as \bar{R} (and hence R) tends to infinity. Therefore we complete Theorem 6.4. \square

Now we can finish the proof of Theorem 6.3.

Proof of Theorem 6.3. Let g be a nonnegative, L^1 -integrable and f -subharmonic function defined on M . Now we define

$$g(x, t) = \int_M H(x, y, t) g(y) d\mu(y)$$

with $g(x, 0) = g(x)$. By Theorem 6.4,

$$\begin{aligned} \frac{\partial}{\partial t} g(x, t) &= \int_M \frac{\partial}{\partial t} H(x, y, t) g(y) d\mu(y) \\ &= \int_M \Delta_f H(x, y, t) g(y) d\mu(y) \\ &= \int_M H(x, y, t) \Delta_f g(y) d\mu(y) \geq 0. \end{aligned}$$

Therefore $g(x, t)$ is increasing in t . Meanwhile, by the mean value theorem, for any $R > 0$ there exists $\bar{R} \in (R, R+1)$ such that

$$\int_{\partial B_o(\bar{R})} |\nabla H|(x, y, t) d\mu_{\sigma, \bar{R}}(y) = \int_{B_o(R+1) \setminus B_o(R)} |\nabla H|(x, y, t) d\mu(y).$$

Since

$$\int_{B_o(\bar{R})} \Delta_f H(x, y, t) d\mu(y) \leq \int_{\partial B_o(\bar{R})} |\nabla H|(x, y, t) d\mu_{\sigma, \bar{R}}(y),$$

so we have

$$\begin{aligned} I &:= \int_{B_o(\bar{R})} \Delta_f H(x, y, t) d\mu(y) \\ &\leq \int_{B_o(R+1) \setminus B_o(R)} |\nabla H|(x, y, t) d\mu(y) \\ &\leq C(n, K, A) \left[V_f^{-1} + t^{-1} V_f^{-\frac{1}{2}} H^{\frac{1}{2}}(x, t) \right]^{1/2} \\ &\quad \times \exp \left[\frac{c_8}{2} K t + \frac{-c_7(R - d(o, x))^2}{2t} \right] \end{aligned}$$

for sufficiently large R , where we used (6.11) for second inequality above. Similar the above arguments, for any a fixed point $x \in M$, if we choose t sufficiently small, then I tends to zero when R and hence \bar{R} tends to infinity. Hence

$$\frac{\partial}{\partial t} \int_M H(x, y, t) d\mu(y) = \int_M \Delta_f H(x, y, t) d\mu(y) = 0,$$

which implies

$$\int_M H(x, y, t) d\mu(y) = 1$$

for all $x \in M$ and for all sufficient small fixed t . Moreover, this equality implies

$$\int_M g(x, t) dx = \int_M \int_M H(x, y, t) g(y) d\mu(y) d\mu(x) = \int_M g(y) dy.$$

Since $g(x, t)$ is increasing in t , we conclude that $g(x, t) = g(x)$ and

$$\begin{aligned} \Delta_f g(x) &= \Delta_f g(x, t) = \int_M \Delta_f H(x, y, t) g(y) d\mu(y) \\ &= g(\xi) \cdot \int_M \Delta_f H(x, y, t) d\mu(y) = 0. \end{aligned}$$

On the other hand, for any positive constant a , we define a new function

$$h(x) := \min\{g(x), a\}.$$

Then it satisfies

$$h(x) \leq g(x), \quad |\nabla h| \leq |\nabla g| \quad \text{and} \quad \Delta_f h(x) \leq 0.$$

In particular, it will satisfy the same estimates, (6.5) and (6.6), as g . Hence we can show that

$$\frac{\partial}{\partial t} \int_M H(x, y, t) h(y) d\mu(y) = \int_M H(x, y, t) \Delta_{f_y} h(y) d\mu(y) \leq 0.$$

Note that h is still L^1 , following the same argument as before, we have $\Delta_f h(x) = 0$. Hence regularity theory asserts that h must be smooth. Since a is arbitrary, this is impossible unless g is identically constant. \square

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